

# Ground state property of one-dimensional Bose-Fermi mixture

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Bose-Fermi mixtures in one dimension are studied in detail on the basis of exact solution. Corresponding to three possible choices of the reference state in quantum inverse scattering method, three sets of Bethe-ansatz equations are derived explicitly. The features of the ground state and low-lying excitations are investigated. The ground-state phase diagram caused by the external field and chemical potential is obtained.

## I. INTRODUCTION

The study on the exactly solvable models has been an important topic for four decades because the perturbative approach is unapplicable for strongly correlated systems. Particles with  $\delta$ -function interaction in one dimension is a simple but interesting model. Since Lieb and Liniger [1] first solved a spinless Bose system with  $\delta$ -function interaction, there has been much more progress [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] in this field. Particularly, when solving the two-component Fermi system, Yang [6] proposed the well known Yang-Baxter equation which has had a significant impacts in both physics and mathematics. As the 2-component system is mostly associated with “spin-1/2” particles that are conventionally referred as Fermi system, the coordinate Bethe ansatz has not been applied to 2-component Bose system till recently [13], which is motivated by spinor Bose-Einstein condensate in magnetically trapped  $^{87}\text{Rb}$  [14].

Recent observation of the superfluid to Mott insulator transition in ultracold atoms in optical lattice [15] stimulated research interests related to strongly correlated atomic gases. Most recently, the physics of ultracold Bose-Fermi mixtures [16] such as  $^7\text{Li}$  -  $^6\text{Li}$  or  $^{87}\text{Rb}$  -  $^{40}\text{K}$  mixtures become a remarkable topics [17, 18, 19, 20, 21]. It is therefore worthwhile to investigate the features of Bose-Fermi system on the basis of exact solutions. The mixed system of bosons and fermions with  $\delta$ -function interaction was discussed earlier in Ref.[10] where the ground state energy and gapless fermionic excitations are calculated in the thermodynamic limit. However, as we aware, the ground state phase diagram under the influence of external field and chemical potential has not been studied yet. These properties become more and more important nowadays due to the rapid progress in the field of cold atomic physics.

In this paper we study one dimensional cold atomic system of Bose-Fermi mixture systematically. Our paper is organized as follows. In the following section we introduce the model and derive its secular equation. In section III, we diagonalize the secular equation by means of QISM method for three cases respectively. Consequently, three different kinds of nested Bethe-ansatz equations are obtained. In section IV, we explicitly analyze the ground state and the possible low-lying excitations. The energy-momentum spectrum for each excitation is calculated numerically from the Bethe-ansatz equation. In the last section, we study the system under the influence of magnetic fields and chemical potentials for the particles to obtain the phase diagram.

## II. THE MODEL AND ITS SECULAR EQUATION

We consider a mixture of cold Bose gas and Fermi gas in one dimension. The Hamiltonian of the system is described by Gross-Pitaevskii functional,

$$\mathbb{H} = \int (\sum_a \partial_x \Psi_a^* \partial_x \Psi_a + c \sum_{a,b} \Psi_a^* \Psi_a \Psi_b^* \Psi_b) dx \quad (1)$$

where the natural unit is adopted for simplicity,  $c$  denotes the interaction strength and  $a, b = 1, 2, 3$  refer to the three components of  $SU(1|2)$  fields. This is an isotropic case of the model considered by Cazallia et al.[22] where approximation method was employed. We just consider this case because the anisotropic case is unintegrable. Among these three fields, two obey anti-commutation relation and one obeys commutation relation. It is convenient to consider the states that span a Hilbert space of  $N$  particles

$$|\psi\rangle = \sum_{x_1, x_2, \dots} \psi_{a_1, \dots, a_N}(x_1, x_2, \dots, x_N) \Psi_{a_1}(x_1) \cdots \Psi_{a_N}(x_N) |0\rangle$$

The eigenvalue problem  $\mathbb{H}|\psi\rangle = E|\psi\rangle$  becomes an  $N$ -particle quantum mechanical problem with the first quantized Hamiltonian,

$$\mathcal{H} = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j) \quad (2)$$

Such a system can be solved by means of Bethe-ansatz approach. Hereby we give a brief description of the main idea of this approach. In the domain  $x_i \neq x_j$ , the Hamiltonian (2) reduces to that for free particles and its eigenfunctions are therefore just superpositions of plane waves. When two particles collide, a scattering process occurs, which is supposed to be a pure elastic process, i.e. exchange of their momenta. So for a given momentum  $k = (k_1, k_2, \dots, k_N)$ , the scattering momenta include all permutations of the components of  $k$ . Because the Hamiltonian is invariant under the action of the permutation group  $S_N$ , one can adopt the following Bethe-ansatz wavefunction

$$\psi_a(x) = \sum_{P \in S_N} A_a(P, Q) e^{i(Pk|Qx)} \quad (3)$$

where  $a = (a_1, a_2, \dots, a_N)$ ,  $a_j$  denotes the  $SU(1|2)$  component of the  $j$ th particles;  $Pk$  denotes the image of a given  $k := (k_1, k_2, \dots, k_N)$  by a mapping  $P \in S_N$ ;  $(Pk|Qx) = \sum_{j=1}^N (Pk)_j(Qx)_j$ . And the coefficients  $A(P, Q)$  are functions of  $P$  and  $Q$  where  $Q$  denotes the permutation such that  $0 < x_{Q_1} < x_{Q_2} < \dots < x_{Q_N} < L$ . For the Bose-Fermi mixture, the wave function should be either symmetric or antisymmetric under permutation  $\Pi^j$  depending on whether they involve Bose label or Fermi labels.

$$(\Pi^j \psi)_a(x) = \pm \psi_{\Pi^j a}(x) \quad (4)$$

The  $\delta$ -function term in the Hamiltonian contributes a boundary condition across the hyper-plane  $x_{Q_j} = x_{Q_{j+1}}$ . Substituting the Beth-Ansatz wave function into this boundary condition and using the continuity condition together with the permutation symmetry, we obtain the following relation

$$A_a(\Pi^j P, Q) = \frac{i[(Pk)_j - (Pk)_{j+1}]P^j + c}{i[(Pk)_j - (Pk)_{j+1}]P^j - c} A_a(P, Q) \quad (5)$$

where  $P^j$  is the permutation between particles at  $x_{Q_j}$  and  $x_{Q_{j+1}}$ , which is given in appendix for concrete choice of the Bose-Fermi labels. For example, if we consider the wave functions of two particles, because of the different exchange symmetries, the wave function of two bosons is  $\frac{1}{\sqrt{2}}(|\psi_1 > |\psi_2 > + |\psi_2 > |\psi_1 >)$ , and  $\frac{1}{\sqrt{2}}(|\psi_1 > |\psi_2 > - |\psi_2 > |\psi_1 >)$

for two fermions. The permutation for two bosons is  $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ , and  $-P$  for two fermions. The matrix

relating to the various amplitudes in the same region given in Eq.(5) is conventionally called  $S$ -matrix

$$S^{j,j+1} = \frac{i[(Pk)_j - (Pk)_{j+1}]P^j + c}{i[(Pk)_j - (Pk)_{j+1}]P^j - c}$$

The amplitudes in region  $Q$  and in its adjacent region  $Q'$  are related by the  $R$ -matrix  $R = PS$ ,

$$A_{a_1 \dots a_i \dots a_j \dots a_N}(Q') = (R^{ij})_{a_1 \dots a_N}^{b_1 \dots b_N} A_{b_1 \dots b_N}(Q) = (R^{ij})_{a_i a_j}^{b_i b_j} A_{a_1 \dots b_i \dots b_j \dots a_N}(Q)$$

If  $x$  is a point in the region  $C(Q^{(i)})$ , then  $x' = (x_1, \dots, x_{Q_1} + L, \dots, x_N)$  is a point in the region  $C(\gamma Q^{(i-1)})$  with  $\gamma = \Pi^{N-1} \Pi^{N-2} \dots \Pi^2 \Pi^1$ . Thus the periodic boundary condition imposes a relation between the wave functions defined on  $C(Q^{(i)})$  and  $C(\gamma Q^{(i-1)})$ . Writing out this relation in terms of Eq.(3), we find that the periodic boundary conditions are guaranteed provided that  $A(P; \gamma Q^{(i-1)}) e^{i(Pk)_1 L} = A(P; Q^{(i)})$ . After applying the  $R$  matrix successively, we obtain the following secular equation,

$$R^{Q^1, Q^N} \dots R^{Q^1, Q^{(i+1)}} R^{Q^1, Q^i} \dots R^{Q^1, Q^2} A(P; Q^{(i)}) = e^{-i(Pk)_1 L} A(P; Q^{(i)}) \quad (6)$$

### III. DIAGONALISATION BY QUANTUM INVERSE SCATTERING METHOD

To determine the spectrum, we should diagonalize the secular equation (6). This can be done by diagonalizing the operator product in the left hand side of Eq. (6), namely, solving the eigenvalues of the operator

$$(T_j)_{a_1 a_2 \dots a_N}^{b_1 b_2 \dots b_N} = (R^{jj-1} \dots R^{j1} R^{jN} \dots R^{jj+1})_{a_1 a_2 \dots a_N}^{b_1 b_2 \dots b_N} \quad (7)$$

where

$$R^{ij} = \frac{(\alpha_i - \alpha_j)I^{ij} - icP^{ij}}{\alpha_i - \alpha_j + ic}. \quad (8)$$

Since they satisfy the Yang-Baxter relation:

$$R^{kj}(\alpha - \beta)R^{ki}(\alpha)R^{ji}(\beta) = R^{ji}(\beta)R^{ki}(\alpha)R^{kj}(\alpha - \beta) \quad (9)$$

the diagonalization can be carried out by means QISM. For Eq. (8), a  $9 \times 9$  monodromy can be defined in the conventional way

$$(T)_{a_1 a_2 \dots a_N, \mu}^{b_1 b_2 \dots b_N, \nu} = \sum_{S_1 S_2 \dots S_{N-1}} (R^{1A})_{a_1, \mu}^{b_1, S_1} (R^{2A})_{a_2, S_1}^{b_2, S_2} \dots (R^{NA})_{a_N, S_{N-1}}^{b_N, \nu} \quad (10)$$

which can be written as a  $3 \times 3$  matrix in the auxiliary space:

$$T = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \quad (11)$$

in which every matrix element is an operator in quantum space. It obeys the following RTT relations,

$$R(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda - \mu) \quad (12)$$

where  $T_1 = T_1 \otimes I$ ,  $T_2 = I \otimes T_2$  with  $I$  the  $3 \times 3$  unitary matrix in quantum space.

Since the Bose-Fermi mixture is  $SU(1|2)$  supersymmetric system, the application of QISM becomes complicated. In the  $SU(3)$  case a unique nested Bethe-ansatz equation was derived [23]. In present case, however, there are three possibilities in choosing the reference state (“pseudo-vacuum”) and the successive orders of the other states, and hence three types of nested Bethe-ansatz equations have to be derived. In the following, we will consider those three cases respectively.

### A. BFF CASE

We first choose the Bose state as the reference state  $|1\rangle$  and the other two states  $|2\rangle$  and  $|3\rangle$  are Fermi states, then the permutation operator is easily written out (see  $P_1$  in (A1)). This case was once noticed by Sutherland [9] in lattice model. Consequently, the RTT relation (12) gives rise to 2 commutation relations between A and B, and 8 commutation relations between B and D. We can write them in a form of tensor product:

$$A(\lambda) \otimes \begin{pmatrix} B_1(\mu) & B_2(\mu) \end{pmatrix} = \frac{k(\mu - \lambda)}{b(\mu - \lambda)} (B_1(\mu) \ B_2(\mu)) \otimes A(\lambda) - \frac{d(\mu - \lambda)}{b(\mu - \lambda)} (B_1(\lambda) \ B_2(\lambda)) \otimes A(\mu) \quad (13)$$

$$\begin{aligned} & \begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \otimes (B_1(\mu) \ B_2(\mu)) = (B_1(\mu) \ B_2(\mu)) \otimes \begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \times \\ & \frac{a(\lambda - \mu)}{b(\lambda - \mu)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{d}{a} & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & -\frac{d}{a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{d(\lambda - \mu)}{b(\lambda - \mu)} (B_1(\lambda) \ B_2(\lambda)) \otimes \begin{pmatrix} D_{11}(\mu) & D_{12}(\mu) \\ D_{21}(\mu) & D_{22}(\mu) \end{pmatrix} \end{aligned} \quad (14)$$

where we used the definition:

$$k(\alpha) = \frac{\alpha - ic}{\alpha + ic}, \quad a(\alpha) = 1, \quad b(\alpha) = \frac{\alpha}{\alpha + ic}, \quad d(\alpha) = \frac{-ic}{\alpha + ic}$$

It is convenient to write the R-matrix in auxiliary space in terms of the generators of the corresponding Lie algebra (see appendix)

$$R(\alpha) = \begin{pmatrix} \frac{\alpha}{\alpha + ic}I - ic\frac{2H_1 + H_2 + I}{3} & \frac{-ic}{\alpha + ic}E_{-\alpha_1} & \frac{-ic}{\alpha + ic}E_{-(\alpha_1 + \alpha_2)} \\ \frac{-ic}{\alpha + ic}E_{\alpha_1} & \frac{\alpha}{\alpha + ic}I + \frac{ic}{\alpha + ic}\frac{H_2 - H_1 + I}{3} & \frac{ic}{\alpha + ic}E_{-\alpha_2} \\ \frac{-ic}{\alpha + ic}E_{(\alpha_1 + \alpha_2)} & \frac{ic}{\alpha + ic}E_{\alpha_2} & \frac{\alpha}{\alpha + ic}I + \frac{ic}{\alpha + ic}\frac{I - H_1 - 2H_2}{3} \end{pmatrix} \quad (15)$$

and to define a pseudo-vacuum as  $N$  particles in the highest weight state of the  $SU(1|2)$  systems, *i.e.*,

$$|\phi\rangle = \prod_{i=1}^N \otimes |1\rangle \quad (16)$$

with  $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  and  $|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . We have  $H_1 |1\rangle = |1\rangle$ ,  $E_{-\alpha_1} |1\rangle = |2\rangle$ ,  $E_{-(\alpha_1+\alpha_2)} |1\rangle = |3\rangle$ , and null if the other operators act on the state  $|1\rangle$ . Thus the vertex (15) becomes triangular when acting on the highest weight state. The eigenvalues of diagonal terms of the monodromy can be obtained as the products of the values in each quantum space, and hence the eigenvalues of the operator  $A(k)$  and  $D(k)$  are:

$$A(k) |\phi\rangle = \prod_{l=1}^N \frac{k - k_l - ic}{k - k_l + ic} |\phi\rangle \quad (17)$$

$$D(k) |\phi\rangle = \prod_{l=1}^N \frac{k - k_l}{k - k_l + ic} |\phi\rangle \quad (18)$$

here

$$D(k) = \begin{pmatrix} D_{11}(k) & D_{12}(k) \\ D_{21}(k) & D_{22}(k) \end{pmatrix} \quad (19)$$

is the monodromy of  $SU(2)$  algebra nested in  $SU(1|2)$ .

Employing the lowering operator  $B(\lambda)$  acting on the pseudo-vacuum, we construct an eigenstate

$$|\omega\rangle = B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|\phi\rangle \quad (20)$$

where  $B(\lambda)$  refers to one of the  $B_1(\lambda)$  or  $B_2(\lambda)$  in the monodromy (11). This state can be used to diagonalize the secular equation. The eigenvalue of the left hand side of the secular equation (6) can be written as the eigenvalue of the trace of the  $SU(1|2)$  monodromy matrix:

$$\begin{aligned} \Lambda_{SU(1|2)}(k; \lambda_1, \lambda_2 \cdots \lambda_M) |\omega\rangle &= Tr(T) |\omega\rangle = (A(k) + tr(D(k))) |\omega\rangle \\ &= \prod_{l=1}^N \frac{k - k_l - ic}{k - k_l + ic} \prod_{\alpha=1}^M \frac{k(\lambda_\alpha - k)}{b(\lambda_\alpha - k)} |\omega\rangle \\ &+ \prod_{l=1}^N \frac{k - k_l}{k - k_l + ic} \prod_{\alpha=1}^M \frac{a(k - \lambda_\alpha)}{b(k - \lambda_\alpha)} \cdot \Lambda'_{SU(2)} |\omega\rangle \end{aligned} \quad (21)$$

From this nested Bethe-ansatz structure we can see that there is a  $SU(2)$  substructure in the  $SU(1|2)$  system when the boson state is chosen as the reference state. In terms of the  $\tilde{R}$ -matrix of  $SU(2)$  appearing in Eq.(14)

$$\tilde{r} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{d}{a} & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & -\frac{d}{a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (22)$$

and the permutation matrix of  $SU(2)$

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (23)$$

we have the  $R$ -matrix  $r = p \cdot \tilde{r}$ . Writing it out in auxiliary space in terms of spin operators in quantum space,

$$r(\alpha) = \begin{pmatrix} \frac{\alpha+ic/2}{\alpha+ic}I + \frac{ic/2}{\alpha+ic}\sigma_z & \frac{ic}{\alpha+ic}\sigma^- \\ \frac{ic}{\alpha+ic}\sigma^+ & \frac{\alpha+ic/2}{\alpha+ic}I - \frac{ic/2}{\alpha+ic}\sigma_z \end{pmatrix}$$

we obtain the fundamental commutation relation from the RTT relations,  $r \cdot \tilde{T}_1 \cdot \tilde{T}_2 = \tilde{T}_2 \cdot \tilde{T}_1 \cdot r$ , as follows

$$\begin{aligned} A'(\lambda)B'(\mu) &= \frac{a'(\mu - \lambda)}{b'(\mu - \lambda)}B'(\mu)A'(\lambda) - \frac{d'(\mu - \lambda)}{b'(\mu - \lambda)}B'(\lambda)A'(\mu) \\ D'(\lambda)B'(\mu) &= \frac{a'(\lambda - \mu)}{b'(\lambda - \mu)}B'(\mu)D'(\lambda) - \frac{d'(\lambda - \mu)}{b'(\lambda - \mu)}B'(\lambda)D'(\mu) \end{aligned} \quad (24)$$

where

$$a'(\alpha) = 1, \quad b'(\alpha) = \frac{b}{a} = \frac{\alpha}{\alpha + ic}, \quad d'(\alpha) = -\frac{d}{a} = \frac{ic}{\alpha + ic}$$

and the  $D(k)$  matrix, a sub-matrix of the  $SU(1|2)$  matrix, is regarded as the  $SU(2)$  monodromy, namely

$$\tilde{T}(k) = \begin{pmatrix} A'(k) & B'(k) \\ C'(k) & D'(k) \end{pmatrix}$$

where  $A'(k) = D_{11}(k)$ ,  $B'(k) = D_{12}(k)$ ,  $C'(k) = D_{21}(k)$ ,  $D'(k) = D_{22}(k)$ .

According to the procedure of QISM [12], the pseudo-vacuum is defined as the product of the highest weight states of  $SU(2)$   $|\phi'\rangle = \prod_{\alpha=1}^M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  which fulfills

$$A'(k)|\phi'\rangle = \prod_{\alpha=1}^M a'(k - \lambda_{\alpha})|\phi'\rangle, \quad D'(k)|\phi'\rangle = \prod_{\alpha=1}^M b'(k - \lambda_{\alpha})|\phi'\rangle$$

In terms of the lowering operator  $B'(k - \lambda_{\alpha})$  in  $SU(2)$  monodromy, one can construct a general state

$$|\omega'\rangle = B'(k - \lambda_1)B'(k - \lambda_2) \cdots B'(k - \lambda_{M'})|\phi'\rangle \quad (25)$$

Using the fundamental commutation relations (24), one obtain that

$$\begin{aligned} \Lambda'_{SU(2)}(k; \mu_1, \mu_2, \cdots \mu_{M'})|\omega'\rangle &= tr(\tilde{T}(k))|\omega'\rangle \\ &= \left( \prod_{\alpha=1}^M a'(k - \lambda_{\alpha}) \prod_{\beta=1}^{M'} \frac{a'(\mu_{\beta} - k)}{b'(\mu_{\beta} - k)} + \prod_{\alpha=1}^M b'(k - \lambda_{\alpha}) \prod_{\beta=1}^{M'} \frac{a'(k - \mu_{\beta})}{b'(k - \mu_{\beta})} \right) |\omega'\rangle \\ &= \left( \prod_{l=\beta}^{M'} \frac{\mu_{\beta} - k + ic}{\mu_{\beta} - k} + \prod_{\alpha=1}^M \frac{k - \lambda_{\alpha}}{k - \lambda_{\alpha} + ic} \prod_{\beta=1}^{M'} \frac{k - \mu_{\beta} + ic}{k - \mu_{\beta}} \right) |\omega'\rangle \end{aligned} \quad (26)$$

The unwanted terms vanishes as long as the following equations hold

$$- \prod_{\beta=1}^{M'} \frac{\mu_{\beta} - \mu_{\beta} - ic}{\mu_{\beta} - \mu_{\beta} + ic} = \prod_{\alpha=1}^M \frac{\mu_{\beta} - \lambda_{\alpha}}{\mu_{\beta} - \lambda_{\alpha} + ic} \quad (27)$$

As a result, Eq. (21) becomes

$$\begin{aligned} \Lambda_{SU(1|2)}(k; \lambda_1, \lambda_2, \cdots \lambda_M) &= \prod_{l=1}^N \frac{k - k_l - ic}{k - k_l + ic} \prod_{\alpha=1}^M \frac{k(\lambda_{\alpha} - k)}{b(\lambda_{\alpha} - k)} + \prod_{l=1}^N \frac{k - k_l}{k - k_l + ic} \prod_{\alpha=1}^M \frac{a(k - \lambda_{\alpha})}{b(k - \lambda_{\alpha})} \cdot \Lambda'_{SU(2)} \\ &= \prod_{l=1}^N \frac{k - k_l - ic}{k - k_l + ic} \prod_{\alpha=1}^M \frac{\lambda_{\alpha} - k - ic}{\lambda_{\alpha} - k} \\ &\quad + \prod_{l=1}^N \frac{k - k_l}{k - k_l + ic} \prod_{\alpha=1}^M \frac{k - \lambda_{\alpha} + ic}{k - \lambda_{\alpha}} \left( \prod_{\beta=1}^{M'} \frac{\mu_{\beta} - k + ic}{\mu_{\beta} - k} + \prod_{\alpha=1}^M \frac{k - \lambda_{\alpha}}{k - \lambda_{\alpha} + ic} \prod_{\beta=1}^{M'} \frac{k - \mu_{\beta} + ic}{k - \mu_{\beta}} \right) \end{aligned} \quad (28)$$

To get rid of the unwanted terms in the expansion, the following equations need to be satisfied

$$1 = - \prod_{l=1}^N \frac{\lambda_\gamma - k_l - ic}{\lambda_\gamma - k_l} \prod_{\beta=1}^{M'} \frac{\mu_\beta - \lambda_\gamma}{\mu_\beta - \lambda_\gamma + ic} \quad (29)$$

It is convenient to redefine the parameter  $\lambda'_\gamma$  by  $\lambda_\gamma - ic/2$ . The equations (29) and (27) for the complete cancellation of the unwanted terms appearing in both procedures, together with the relation resulting from periodic boundary conditions,  $e^{-ik_\alpha L} = \Lambda_{SU(1|2)}(k; \lambda_1, \lambda_2, \dots, \lambda_M)$  gives rise to the Bethe-Ansatz equations

$$\begin{aligned} e^{ik_j L} &= - \prod_{l=1}^N \frac{k_j - k_l + ic}{k_j - k_l - ic} \prod_{\alpha=1}^M \frac{k_j - \lambda_\alpha - ic/2}{k_j - \lambda_\alpha + ic/2} \\ 1 &= - \prod_{l=1}^N \frac{\lambda_\gamma - k_l - ic/2}{\lambda_\gamma - k_l + ic/2} \prod_{\beta=1}^{M'} \frac{\lambda_\gamma - \mu_\beta + ic/2}{\lambda_\gamma - \mu_\beta - ic/2} \\ 1 &= - \prod_{\alpha=1}^M \frac{\mu_c - \lambda_\alpha - ic/2}{\mu_c - \lambda_\alpha + ic/2} \prod_{\beta=1}^{M'} \frac{\mu_c - \mu_\beta + ic}{\mu_c - \mu_\beta - ic} \end{aligned} \quad (30)$$

which determine the spectrum of the  $SU(1|2)$  system.

## B. FBF CASE

We now consider the second case, in which the Bose state is chosen as the second state, while the first and third are Fermi state. From the permutation operator  $P_2$  (see (A2)), we can get R-matrix (8). Using the same monodromy as in Eq. (11) and the RTT relation(12), we get the following commutation relations:

$$\begin{aligned} A(\lambda) \otimes (B_1(\mu) \ B_2(\mu)) &= \frac{a(\mu - \lambda)}{b(\mu - \lambda)} (B_1(\mu) \ B_2(\mu)) \otimes A(\lambda) \\ &\quad - \frac{d(\mu - \lambda)}{b(\mu - \lambda)} (B_1(\lambda) \ B_2(\lambda)) \otimes A(\mu) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (31)$$

$$\begin{aligned} \begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \otimes (B_1(\mu) \ B_2(\mu)) \\ &= (B_1(\mu) \ B_2(\mu)) \otimes \begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \frac{k(\lambda - \mu)}{b(\lambda - \mu)} \cdot \check{r} \\ &\quad - \frac{d(\lambda - \mu)}{b(\lambda - \mu)} (B_1(\lambda) \ B_2(\lambda)) \otimes \begin{pmatrix} D_{11}(\mu) & D_{12}(\mu) \\ D_{21}(\mu) & D_{22}(\mu) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (32)$$

In this case, we can see that there is a nested  $SU(1|1)$  substructure in the  $SU(1|2)$  system.  $\check{R}$ -matrix of  $SU(1|1)$  appeared in Eq.(32) reads

$$\check{r} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{d}{k} & \frac{b}{k} & 0 \\ 0 & \frac{b}{k} & \frac{d}{k} & 0 \\ 0 & 0 & 0 & \frac{a}{k} \end{pmatrix} \quad (33)$$

In this  $SU(1|1)$  substructure, the Bose state is chosen as the highest weight state  $|\phi'\rangle = \prod_{i=1}^M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  when the QISM [12]

is applied. We can obtain the Bethe-Ansatz equation by the similar procedure applied in previous case,

$$\begin{aligned}
e^{ik_j L} &= - \prod_{\alpha=1}^M \frac{k_j - \lambda_\alpha + ic/2}{k_j - \lambda_\alpha - ic/2} \\
1 &= - \prod_{l=1}^N \frac{\lambda_\gamma - k_l - ic/2}{\lambda_\gamma - k_l + ic/2} \prod_{\beta=1}^{M'} \frac{\mu_\beta - \lambda_\gamma - ic/2}{\mu_\beta - \lambda_\gamma + ic/2} \\
1 &= - \prod_{\alpha=1}^M \frac{\mu_c - \lambda_\alpha + ic/2}{\mu_c - \lambda_\alpha - ic/2}
\end{aligned} \tag{34}$$

### C. FFB CASE

We turn to the case that the Bose state is chosen as the third state, and one of the Fermi state as the reference state, the other Fermi state as the second state. In terms of the permutation matrix  $P_3$  (see(A3)), we can get the R-matrix(8). The RTT relation(12) gives rise to the following communication relations:

$$\begin{aligned}
A(\lambda) \otimes (B_1(\mu) \ B_2(\mu)) &= \frac{a(\mu - \lambda)}{b(\mu - \lambda)} (B_1(\mu) \ B_2(\mu)) \otimes A(\lambda) \\
&\quad - \frac{d(\mu - \lambda)}{b(\mu - \lambda)} (B_1(\lambda) \ B_2(\lambda)) \otimes A(\mu) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \otimes (B_1(\mu) \ B_2(\mu)) \\
&= (B_1(\mu) \ B_2(\mu)) \otimes \begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \frac{a(\lambda - \mu)}{b(\lambda - \mu)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{d}{a} & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & \frac{d}{a} & 0 \\ 0 & 0 & 0 & \frac{k}{a} \end{pmatrix} \\
&\quad - \frac{d(\lambda - \mu)}{b(\lambda - \mu)} (B_1(\lambda) \ B_2(\lambda)) \otimes \begin{pmatrix} D_{11}(\mu) & D_{12}(\mu) \\ D_{21}(\mu) & D_{22}(\mu) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{36}$$

They are almost the same as that in the “FBF” case, there is also a nested  $SU(1|1)$  substructure whose  $\tilde{r}$  matrix appears in Eq.(36). The following nested Bethe-ansatz equation is obtained

$$\begin{aligned}
e^{ik_j L} &= - \prod_{\alpha=1}^M \frac{k_j - \lambda_\alpha + ic/2}{k_j - \lambda_\alpha - ic/2} \\
1 &= - \prod_{l=1}^N \frac{\lambda_\gamma - k_l - ic/2}{\lambda_\gamma - k_l + ic/2} \prod_{\alpha=1}^M \frac{\lambda_\gamma - \lambda_\alpha + ic}{\lambda_\gamma - \lambda_\alpha - ic} \prod_{\beta=1}^{M'} \frac{\mu_\beta - \lambda_\gamma + ic/2}{\mu_\beta - \lambda_\gamma - ic/2} \\
1 &= - \prod_{\alpha=1}^M \frac{\mu_c - \lambda_\alpha - ic/2}{\mu_c - \lambda_\alpha + ic/2}
\end{aligned} \tag{37}$$

which were also derived in Ref. [10] by means of the coordinate Bethe ansatz.

## IV. GROUND STATE AND ITS LOW-LYING EXCITATIONS

### A. BFF CASE

In this case, the boson state is chosen as the reference state. there are  $N$  particles in all. After  $M$  lower-operators  $B$  act on the reference state, there just  $N - M$  bosons. Analogously, after  $M'$  lower-operators act on the second state,

there are  $M - M'$  fermions of species 1 and  $M'$  fermions of species 2. The same analysis in other two case. Taking the logarithm of (30), we have

$$\begin{aligned}
k_j L &= 2\pi I_j + \sum_{l=1}^N \Theta_1(k_j - k_l) + \sum_{\alpha=1}^M \Theta_{-1/2}(k_j - \lambda_\alpha) \\
2\pi J_\gamma &= \sum_{l=1}^N \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\beta=1}^{M'} \Theta_{1/2}(\lambda_\gamma - \mu_\beta) \\
2\pi J'_c &= \sum_{\alpha=1}^M \Theta_{-1/2}(\mu_c - \lambda_\alpha) + \sum_{\beta=1}^{M'} \Theta_1(\mu_c - \mu_\beta)
\end{aligned} \tag{38}$$

where  $\Theta_n(x) = -2 \tan^{-1}(x/nc)$  and  $I_j$  is an integer (half-odd integer) if  $N - M - 1$  is even (odd), while  $J_\gamma$  is an integer (half-odd integer) if  $N - M' - 1$  is even (odd), and  $J'_c$  is an integer (half-odd integer) if  $M - M' - 1$  is even (odd). Once all roots  $\{k_j, \lambda_\gamma, \mu_c\}$  are solved from the above equations (38) for a given set of quantum numbers  $\{I_j, J_\gamma, J'_c\}$ , the energy and the momentum will be calculated by

$$E = \sum_{j=1}^N k_j^2, P = \frac{2\pi}{L} \left[ \sum_{j=1}^N I_j - \sum_{\gamma=1}^M J_\gamma - \sum_{c=1}^{M'} J'_c \right]. \tag{39}$$

### 1. The ground state

It is easy to show that the right-hand side of the first equation of equations(38) is a monotonically increasing function of  $k_j$ , i.e. if  $I_i < I_j$ , then  $k_i < k_j$ . Thus the configuration of  $\{I_j\}$  for the ground state is given by successive integers of half-integers symmetrically arranged around zero. Given a set of quantum numbers  $I_j, J_\gamma, J'_c$  with the solutions  $k_j, \lambda_\gamma, \mu_c$ , it is useful to consider the weak-coupling limit  $c \rightarrow 0^+$ . Due to  $\Theta_{\pm n}(x) \rightarrow \mp \pi \text{sgn}(x)$ , Eqs.(38) become

$$\begin{aligned}
2\pi I_j &= k_j L + \sum_{l=1}^N \pi \text{sgn}(k_j - k_l) - \sum_{\alpha=1}^M \pi \text{sgn}(k_j - \lambda_\alpha) \\
2\pi J_\gamma &= \sum_{l=1}^N \pi \text{sgn}(\lambda_\gamma - k_l) - \sum_{\beta=1}^{M'} \pi \text{sgn}(\lambda_\gamma - \mu_\beta) \\
2\pi J'_c &= \sum_{\alpha=1}^M \pi \text{sgn}(\mu_c - \lambda_\alpha) - \sum_{\beta=1}^{M'} \pi \text{sgn}(\mu_c - \mu_\beta)
\end{aligned} \tag{40}$$

The subscripts of the rapidities  $k_j, \lambda_\gamma, \mu_c$  are chosen in such a way that their quantum numbers  $k_j, \lambda_\gamma, \mu_c$  are all ranged in an increasing order. Then we have

$$\begin{aligned}
2(I_{j+1} - I_j - 1) &= (k_{j+1} - k_j) \frac{L}{\pi} - \sum_{\alpha=1}^M [\text{sgn}(k_{j+1} - \lambda_\alpha) - \text{sgn}(k_j - \lambda_\alpha)] \\
2(J_{\gamma+1} - J_\gamma) &= \sum_{l=1}^N [\text{sgn}(\lambda_{\gamma+1} - k_l) - \text{sgn}(\lambda_\gamma - k_l)] - \sum_{\beta=1}^{M'} [\text{sgn}(\lambda_{\gamma+1} - \mu_\beta) - \text{sgn}(\lambda_\gamma - \mu_\beta)] \\
2(J'_{c+1} - J'_c + 1) &= \sum_{\alpha=1}^M [\text{sgn}(\mu_{c+1} - \lambda_\alpha) - \text{sgn}(\mu_c - \lambda_\alpha)]
\end{aligned} \tag{41}$$

Thus, if  $J'_{c+1} - J'_c = m$ , there will be  $m + 1$  solutions of  $\lambda_\alpha$  between  $\mu_c$  and  $\mu_{c+1}$ , and if  $I_{j+1} - I_j = n$ , correspondingly with a  $\lambda_\alpha$  satisfying  $k_j < \lambda_\alpha < k_{j+1}$ , then we will get  $k_{j+1} - k_j = \frac{2\pi n}{L}$ , comparing with no such a  $\lambda_\alpha$  there will be  $k_{j+1} - k_j = \frac{2\pi(n-1)}{L}$ . Obviously, such a  $\lambda_\alpha$  always repels the  $k$  rapidity away, then leading to the rising of the energy. Thus the ground state of this system should be with no  $\lambda_\alpha$  lying in  $k_j$ .



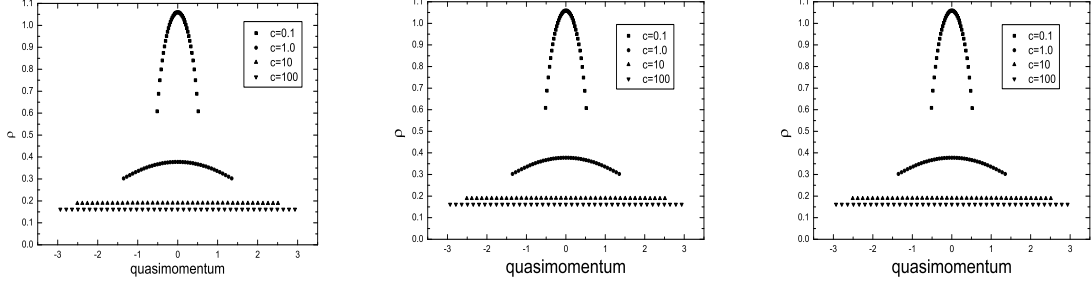


FIG. 1: The density of state in  $k$ -space for the ground state(left), adding one fermion(middle), and adding three fermions(right). The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for  $c = 100, 10, 1, 0.1$ .

In the strong coupling limit  $c \rightarrow \infty$ , we have  $\tan^{-1} \frac{x}{c} \simeq \frac{x}{c}$ . Substituting these to the secular equations (38) for the ground state ( $M = 0, M' = 0$ ) and the low-lying excited state with ( $M = 1, M' = 0$ ), the secular equations become:

$$\begin{aligned} k_j L &= 2\pi I_j - 2 \sum_{l=1}^N \frac{k_j - k_l}{c} \\ k'_j L &= 2\pi I'_j - 2 \sum_{l=1}^N \frac{k'_j - k_l}{c} + 2 \frac{k'_j - \lambda_1}{c/2} \end{aligned}$$

Here we change the ground state by adding one  $\lambda_\alpha$  which leads to  $I_j - I'_j = 1/2$ . From the two equations above we can get:

$$\begin{aligned} (k_{j+1} - k_j) L \left[ 1 + \frac{2N}{cL} \right] &= 2\pi \\ (k'_{j+1} - k'_j) L \left[ 1 + \frac{2N-4}{cL} \right] &= 2\pi \end{aligned} \quad (42)$$

In order to analyzing the low-lying excited characters of the system more conveniently, we introduce density of roots

$$\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)} \quad , \quad \sigma(\lambda_\gamma) = \frac{1}{L(\lambda_{\gamma+1} - \lambda_\gamma)} \quad , \quad \omega(\mu_c) = \frac{1}{L(\mu_{c+1} - \mu_c)}$$

In thermodynamics limit, we have  $\rho(k) = \frac{1}{L} \frac{dI(k)}{dk}$ , corresponding to  $\sigma(\lambda) = \frac{1}{L} \frac{dJ(\lambda)}{d\lambda}$  and  $\omega(\mu) = \frac{1}{L} \frac{dJ'(\mu)}{d\mu}$ . In terms of these densities, the energy and the momentum per length are given by

$$E/L = \int k^2 \rho(k) dk, \quad P/L = \int k \rho(k) dk \quad (43)$$

while  $N$ ,  $M$  and  $M'$  are determined by

$$N/L = \int \rho(k) dk \quad , \quad M/L = \int \sigma(\lambda) d\lambda \quad , \quad M'/L = \int \omega(\mu) d\mu \quad (44)$$

where  $K_n(x) = \frac{1}{\pi} \frac{nc/2}{n^2 c^2/4 + x^2}$  the density of the state satisfies the integral equation

$$\rho_0(k) = \frac{1}{2\pi} + \int_{-k_F}^{k_F} K_2(k - k') \rho_0(k') dk' \quad (45)$$

in the thermodynamic limit, where  $\rho_0(k)$  and  $k_F$  are the density and integration limit for the ground state, respectively. We solved the secular equation for 42 particles with  $M = M' = 0$  numerically, and the density of the ground state is depicted in Fig.1(left) for different coupling constants.

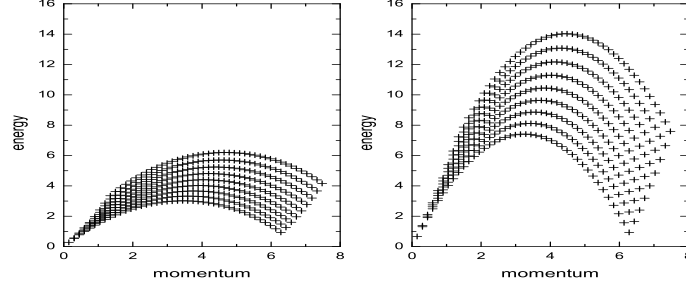


FIG. 2: Particle-hole excitation for  $c = 1.0$  (left) and  $c = 10.0$  (right).

Comparing with the ground state, we plot the spectrum for the low-lying excitation ( $M = 1, M' = 0$ ) in Fig.1(middle). The density of state are slightly compressed compared with Fig.1(middle). It is not obvious in numerical results in Fig.1(middle) comparing to Fig.1(left), so we increase  $M$  from 1 to 3, and the curves are depressed more evidently in the system with  $N = 42$  (Fig.1(right)). As the value of  $M$  rises, the number of fermions rises correspondingly, and the larger the number of the fermions is, the higher the energy should be. As a result, the ground state contains only bosons which agrees with the results of our asymptotic analysis.

### 2. Particle-hole excitation

The quantum numbers for the ground state in  $N$  particles system are  $\{I_j\} = \{-(N-1)/2, \dots, (N-1)/2\}$ ,  $\{J_\gamma\} = \{J'_c\} = \text{empty}$ . If we add a hole to the ground state, then the quantum numbers take the values  $I_1 = -(N-1)/2 + \delta_{1,j}$  for  $(1 \leq j_1 \leq N)$ ,  $I_j = I_{j-1} + \delta_{j,j_1}$  for  $(j = 2, \dots, N-1)$ ,  $I_N = I_0$ , ( $I_0 > (N-1)/2$ ), we call it the particle-hole excitation. In Fig. 2 the excitation spectrum is plotted with coupling numbers ( $c = 1.0, 10.0$ ).

In the thermodynamic limit, we use the expression  $\rho(k) = \rho_0(k) + \rho_1(k)/L$ , then removing one  $I$  from the original symmetric sequence and adding a new  $I_n$  outside it, we have

$$\rho_1(k) + \delta(k - \bar{k}) = \int K_2(k - \bar{k}) \rho_1(k') dk' + K_2(k - k_p) \quad (46)$$

The excited energy consists of two terms  $\Delta E = \frac{1}{L} \int \rho_1(k) k^2 dk + \frac{1}{L} k_p^2 = \xi_h(\bar{k}) + \xi_a(k_p)$ , where  $\xi_h$  is holon's energy and  $\xi_a(k_p)$  is particle's energy, and they can be calculated by

$$\begin{aligned} \rho_1^h(k, \bar{k}) &= \int K_2(k - k') \rho_1^h(k' - \bar{k}) dk' - K_2(k - \bar{k}) \\ \xi_h(\bar{k}) &= -\bar{k}^2 + \int_{-k_F}^{k_F} k^2 \rho_1^h(k, \bar{k}) dk \end{aligned} \quad (47)$$

### 3. Add one fermion

If we add one fermion into the ground state, this excitation can be characterized by moving the quantum number  $J_1$  in the following region:

$$-(N-1)/2 < J_1 < (N-1)/2$$

We describe this phenomenon in Fig.3(a). Replacing one boson by one fermion corresponds to a two-parameter excitation. Its energy is given by  $\Delta E = \int k^2 \rho_1^c(k, \lambda) dk$  with  $\rho_1(k)$  solving

$$\rho_1(k) + \delta(k - \bar{k}) = \int K_2(k - k') \rho_1(k') dk' + K_1(k - \lambda_1) \quad (48)$$

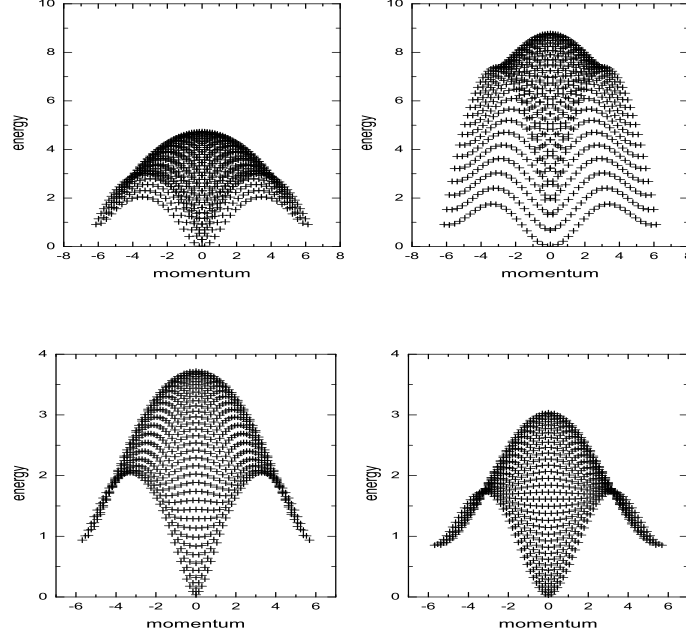


FIG. 3: Top panels show the one-fermion excitation for  $c = 1.0$  (left) and  $c = 10.0$  (right); Bottom panels show the two-fermion excitation for  $c = 1.0$  (left) and  $c = 10.0$  (right).

Then we have one fermion excitation  $\Delta E = \xi_h(\bar{k}) + \xi_c(\lambda)$ , where  $\xi_h(\bar{k})$  is the same to Eqs.(47) and  $\xi_c$  is defined by  $\xi_c(\lambda) = \int k^2 \rho_1^c(k, \lambda) dk$  with

$$\rho_1^c(k, \lambda) = \int_{-k_F}^{k_F} K_2(k - k_l) \rho_1^c(k_l, \lambda) dk_l - K_1(k - \lambda_1) \quad (49)$$

#### 4. Two fermions excitation

If two spin up fermions or two spin down fermions are permitted in this system, then the arrangement of their quantum numbers is

$$-(N-1)/2 \leq J_1 \leq J_2 \leq (N-1)/2$$

Numerical calculation for this type of excitation is shown in Fig.3(b). Furthermore, if we introduce one spin-up and one spin-down fermions in this system, the results are the same as Fig.3(b). Comparing to the system of pure bosons, we found that such excitation likes isospinon-isospion excitation in two-band  $SU(2)$  system[24].

### B. FBF CASE

From Eqs.(34), we know there are  $N - M$  fermions of species 1,  $M - M'$  bosons and  $M'$  fermions of species 2. Taking the logarithm of Eqs.(34), we get

$$\begin{aligned}
k_j L &= 2\pi I_j + \sum_{\alpha=1}^M \Theta_{1/2}(k_j - \lambda_\alpha) \\
2\pi J_\gamma &= \sum_{l=1}^N \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\beta=1}^{M'} \Theta_{1/2}(\lambda_\gamma - \mu_\beta) \\
2\pi J'_c &= \sum_{\alpha=1}^M \Theta_{-1/2}(\mu_c - \lambda_\alpha)
\end{aligned} \tag{50}$$

The quantum number  $I_j$  take integer or half-integer values, depends on whether  $M$  is even or odd. And  $J'_c$  take half-integer(integer) values when  $M$  is even(odd). While  $J_\gamma$  is integer (half-integer), if  $N - M' - 1$  is even (odd). In the weak-coupling limit,  $c \rightarrow 0^+$ ,  $\Theta_{1/2}(x) \rightarrow -\pi \operatorname{sgn}(x)$ , and  $\Theta_{-1/2}(x) \rightarrow \pi \operatorname{sgn}(x)$  for  $x \gg 1$ , hence Eqs.(50) becomes

$$\begin{aligned}
2I_j &= \frac{k_j L}{\pi} + \sum_{\alpha=1}^M \operatorname{sgn}(k_j - \lambda_\alpha) \\
2J_\gamma &= \sum_{l=1}^N \operatorname{sgn}(\lambda_\gamma - k_l) - \sum_{\beta=1}^{M'} \operatorname{sgn}(\lambda_\gamma - \mu_\beta) \\
2J'_c &= \sum_{\alpha=1}^M \pi \operatorname{sgn}(\mu_c - \lambda_\alpha) - \sum_{\beta=1}^{M'} \operatorname{sgn}(\mu_c - \mu_\beta)
\end{aligned} \tag{51}$$

As choosing  $I_j, J_\gamma, J'_c$  in an increasing order, for a given  $M$  and  $M \leq N$  with the rules of Young tableau, the minimum value of the left-hand side of the third equation of Eqs. (51) is  $-M + 2$ . Therefore, the smallest  $\lambda_\alpha$  must be smaller than the smallest  $\mu_c$ . Otherwise the left-hand side would be  $-N$ , and if we take the maximum value of the left-hand side  $M - 2$ , correspondingly the largest  $\lambda_\alpha$  must be larger than the larger  $\mu_c$ . In other words, the presence of  $\mu_c$  is only allowed in  $\lambda$  space. Furthermore, we can obtain  $|J_c| \leq (N - M')/2$ , when there is no spin down fermion ( $M' = 0$ ), all  $J_\gamma$  like to stay in the  $I_j$  sequence.

Supposing the rapidities  $k_j, \lambda_\gamma, \mu_\beta$  are in an increasing order as  $I_j, J_\gamma, J'_c$ , then we have

$$\begin{aligned}
2(I_{j+1} - I_j) &= \frac{(k_{j+1} - k_j)L}{\pi} - \sum_{\alpha=1}^M [\operatorname{sgn}(k_{j+1} - \lambda_\alpha) - \operatorname{sgn}(k_j - \lambda_\alpha)] \\
2(J_{\gamma+1} - J_\gamma) &= \sum_{l=1}^N [\operatorname{sgn}(\lambda_{\gamma+1} - k_l) - \operatorname{sgn}(\lambda_\gamma - k_l)] - \sum_{\beta=1}^{M'} [\operatorname{sgn}(\lambda_{\gamma+1} - \mu_\beta) - \operatorname{sgn}(\lambda_\gamma - \mu_\beta)] \\
2(J'_{c+1} - J'_c) &= \sum_{\alpha=1}^M [\operatorname{sgn}(\mu_{c+1} - \lambda_\alpha) - \operatorname{sgn}(\mu_c - \lambda_\alpha)]
\end{aligned} \tag{52}$$

Therefore, let  $I_{j+1} - I_j = n$  and there will be  $k_{j+1} - k_j = 2\pi(n - 1)/L$  existing a  $\lambda_\alpha$  between  $k_j$  and  $k_{j+1}$ . Otherwise  $k_{j+1} - k_j = 2\pi n/L$ . That means existing a  $\lambda_\alpha$  in  $k_j$  space will decrease the system's energy, so the ground state should have more  $\lambda_\alpha$ . If  $J'_{c+1} - J'_c = m$ , we know there will be  $m$  solutions  $\lambda_\alpha$  satisfying  $\mu_c < \lambda_\alpha < \mu_{c+1}$ . When  $J_{\gamma+1} - J_\gamma = m'$ ,  $m'$  is the integer which equals or surpasses one, then there must be some  $k_j$  lying in neighboring  $\lambda_\gamma$ , and if we have the larger the number of  $\mu_\beta$  between  $\lambda_\gamma$  and  $\lambda_{\gamma+1}$ , the larger the number of  $k_j$  we hope. If  $m'$  is large enough, we wish that  $k_j$  will be large enough, and  $\mu_\beta$  will be small enough. When  $m'$  equals one, there is only one  $k_j$  between  $\lambda_\gamma$  and  $\lambda_{\gamma+1}$ , this state contains no  $\mu_\beta$ . That is to say  $k$  and  $\lambda$  alternate.

Then in the strong-coupling limit  $c \rightarrow \infty$ ,  $\tan^{-1}(x/c) \rightarrow x/c$  and Eqs.(50) give rise to

$$k_j L = 2\pi I_j - 2M \frac{k_j - \lambda_\alpha}{c/2}$$

Furthermore, the new form will be

$$(k_{j+1} - k_j) L \left[ 1 + \frac{4M}{cL} \right] = 2\pi \quad (53)$$

From the formula above, we know if  $M$  approaches to  $N$ ,  $\Delta k = k_{j+1} - k_j$  is the smallest, so does the energy. Therefore, the  $M = N$  state is the ground state.

We describe the density of ground state by numerical approaches, and find that is the same as Fig.1(left). If we permit  $M = N - 3$ , i.e. there are three spin up fermions lying in the ground state, the density of state is also shown in Fig.1(right). It is obvious that the more the fermions lying in this system, the higher the energy will be. Thus the ground state is exactly all bosons without one fermion, which verifies the analysis in case 1. Furthermore we remove one of the  $I$ 's from the ground state sequence and add a "new"  $I_0$  outside. Excited states are obtained by varying the quantum number as

$$I_j = -(N-1)/2, \dots, -(N-1)/2 + i - 1, -(N-1)/2 + i + 1, \dots, (N-1)/2, I_0,$$

$$J_\gamma = -(N-1)/2, \dots, (N-1)/2,$$

consequently we get the excitation spectrum which isn't different from Fig.2.

For this case the excitation of adding one fermion is obtained from  $M = N - 1$ , which contains one free parameter in the  $I$ -sequence and one free parameter in the  $J$ -sequence. Thus the order of quantum numbers is:

$$\begin{aligned} I_1 &= -N/2 + 1 + \delta_{j_1,1} \\ I_j &= I_{j-1} + 1 + \delta_{j_1,j} \quad (j = 2, \dots, N) \\ J_1 &= -N/2 + \delta_{\alpha_1,1}, \\ J_\alpha &= J_{\alpha-1} + 1 + \delta_{\alpha,\alpha_1} \quad (\alpha = 2, \dots, M) \end{aligned}$$

where  $1 \leq j_1 \leq N + 1, 1 \leq \alpha_1 \leq M + 1$ . The excitation spectrum is shown in Fig.3(a), which is also consistent with the excitation of adding one fermion in case 1.

Comparing to the case 1, when we add two spin up fermions ( $M = N - 2$ ), there are two free parameters in the  $J$ -sequence. The result is depicted in Fig.3(b).

### C. FFB CASE

There are  $N - M$  fermions of species 1,  $M - M'$  fermions of species 2 and  $M'$  bosons in Eqs.(37). Also taking the logarithm of these equations we obtain

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_{\alpha=1}^M \Theta_{1/2}(k_j - \lambda_\alpha) \\ 2\pi J_\gamma &= \sum_{l=1}^N \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\alpha=1}^M \Theta_1(\lambda_\gamma - \lambda_\alpha) + \sum_{\beta=1}^{M'} \Theta_{-1/2}(\lambda_\gamma - \mu_\beta) \\ 2\pi J'_c &= \sum_{\alpha=1}^M \Theta_{-1/2}(\mu_c - \lambda_\alpha) \end{aligned} \quad (54)$$

where  $I_j$  is an integer (half-odd integer) if  $N - M - 1$  is even (odd),  $J_\gamma$  is an integer (half-odd integer) if  $N - M - M' - 1$  is even (odd), and  $J'_c$  is an integer (half-odd integer) if  $M - 1$  is even (odd).

Considering the weak coupling limit  $c \rightarrow 0^+$ , we have

$$\begin{aligned}
2(I_{j+1} - I_j) &= \frac{(k_{j+1} - k_j)L}{\pi} + \sum_{\alpha=1}^M [\text{sgn}(k_{j+1} - \lambda_\alpha) - \text{sgn}(k_j - \lambda_\alpha)] \\
2(J_{\gamma+1} - J_\gamma + 1) &= \sum_{l=1}^N [\text{sgn}(\lambda_{\gamma+1} - k_l) - \text{sgn}(\lambda_\gamma - k_l)] \\
&\quad + \sum_{\beta=1}^{M'} [\text{sgn}(\lambda_{\gamma+1} - \mu_\beta) - \text{sgn}(\lambda_\gamma - \mu_\beta)] \\
2(J'_{c+1} - J'_c) &= \sum_{\alpha=1}^M [\text{sgn}(\mu_{c+1} - \lambda_\alpha) - \text{sgn}(\mu_c - \lambda_\alpha)]
\end{aligned} \tag{55}$$

If we set  $J'_{c+1} - J'_c = m$ , there will be  $m$   $\lambda'_\alpha$ s satisfying  $\mu_c < \lambda_\alpha < \mu_{c+1}$ . Letting  $I_{j+1} - I_j = n$ , if there is a  $\lambda_\alpha$  satisfying  $k_j < \lambda_\alpha < k_{j+1}$ , we will get  $k_{j+1} - k_j = 2\pi(n-1)/L$ , otherwise  $k_{j+1} - k_j = 2\pi n/L$ . From those above, we find that adding a  $\lambda_\alpha$  into the  $k_j$  space will expand the distance of neighboring particles and lead to decrease of the energy. Thus the more  $\lambda_\alpha$  lying between  $k_j$  and  $k_{j+1}$ , the lower the energy will be. Letting  $J_{\gamma+1} - J_\gamma = n'$ , there will be  $n' + 1$  solutions of  $k_l$  and  $\mu_\beta$  satisfying  $\lambda_\gamma < k_l$ ,  $\mu_\beta < \lambda_{\gamma+1}$ , where  $n'$  is the integer which equals or surpasses one, i.e.  $n' = 1$ . Then there will be two fermions or two bosons or one fermion and one boson between  $\lambda_\gamma$  and  $\lambda_{\gamma+1}$ . As we always set the distance between the neighboring quantum numbers as one, the most reasonable state should be one fermion and one boson between  $\lambda_\gamma$  and  $\lambda_{\gamma+1}$ .

Therefore, the state of  $M = N, M' = N$  is a boson state which has the lowest energy. This result coincides with parts of Lai's [10] results. The density of the ground state is the same as Fig.1(left). When  $M' = N - 1, M = N$ , i.e. there is one spin-up fermion in the ground state, we have the same result as Fig.1(middle). And the more spin up fermions there are, the higher the energy should be. The particle-hole excitation spectrum is as Fig.2. Due to the restriction given by the Young tableau,  $M = N - 2$  gives the excitation of adding one spin-up and one spin-down fermion, and exactly there are three holes in the  $J$ -sequence. This excitation is depicted in Fig.3(b). It also sustains the analysis of the case 1.

#### D. The consistency of the three cases

In our analytic and numerical results, the density of ground state (Fig.1(left)) and the energy-momentum spectrum of low-lying excitations (Fig.3(a), Fig.3(b)) in the three cases are the same, and the ground states for all the three cases are the states with merely bosons. i.e. in the BFF case, the ground state is " $M = M' = 0$ ", " $M = N, M' = 0$ " for the FBF case and " $M = M' = N$ " for the FFB case, they are all merely boson state. Thus we conclude that the properties of this multi-component system in one dimension are independent of the reference state which we choose, which manifests the consistency of these cases. Such kind of equivalence was noticed in the so called super-symmetry t-J model [25].

### V. THE GROUND STATE PHASE DIAGRAM IN THE PRESENCE OF EXTERNAL MAGNETIC FIELDS

We take into account both the chemical potentials and the external magnetic fields which bring about the Zeeman splitting for fermions. At first, we analyze two limit cases: the case of free particles and the case of particles with infinitely strong interaction. We have got the phase diagram as Fig.4 and Fig.5.

#### A. Weak-coupling limit

When  $c \rightarrow 0$ , we can treat these particles as free particles. Thus in the ground state, only fermions have non-zero momentum because of the pauli excluding principle. We plot the phase diagram in Fig.4 for 42 particles. The region "S" is composed by some rhombuses that stands for different number of Fermi pairs(1, 3, 5, 7, ...). We take two rhombuses (3 and 5 pairs of fermions) for example. In the valley region between them (here we consider  $h > 0$ ), there are 5 spin up and 3 spin down fermions. Thus on the left boundary of this valley, the energy of 3 pairs of spin singlet

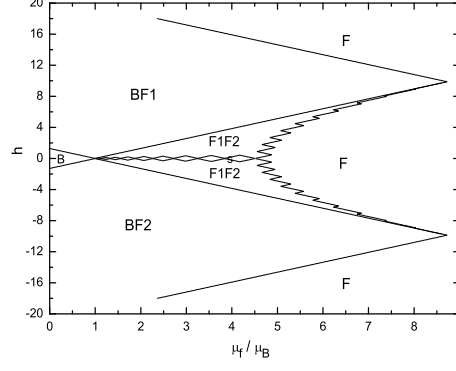


FIG. 4: The phase diagram of free particles. The vertical axis stands for the external magnetic field, and the horizontal axis stands for the ratio of chemical potential between boson ( $\mu_b$ ) and fermion ( $\mu_f$ ). The region “B” stands for the state with merely bosons. “BF1” means the mix state of bosons and the spin up fermions. “BF2” means the mix state of bosons and the spin down fermions. “F1F2” stands for the mix state of bosons and two kinds of fermions, but  $N_\uparrow \neq N_\downarrow$ . The region “S” means the spin singlet state, i.e.  $N_\uparrow = N_\downarrow$ . The region “F” means there is no bosons in ground state.

fermions equals that of 5 spin up plus 3 spin down fermions. Because there are  $N - M$  bosons,  $M - M'$  spin-up fermions, and  $M'$  spin-down fermions,

$$\begin{aligned} H_{3\uparrow 3\downarrow} &= E_{33} - \mu_B \left( (N - M) + \frac{\mu_f}{\mu_B} M \right) - \frac{h}{2} (M - 2M') \\ &= 2 \left( \frac{L}{2\pi} \right)^2 - \mu_B (36 + 6 \frac{\mu_f}{\mu_B}) \end{aligned} \quad (56)$$

Analogously,

$$\begin{aligned} H_{5\uparrow 3\downarrow} &= E_{53} - \mu_B \left( (N - M) + \frac{\mu_f}{\mu_B} M \right) - \frac{h}{2} (M - 2M') \\ &= 12 \left( \frac{L}{2\pi} \right)^2 - \mu_B (34 + 8 \frac{\mu_f}{\mu_B}) - h \end{aligned} \quad (57)$$

Thus the phase boundary can be described as  $H_{3\uparrow 3\downarrow} = H_{5\uparrow 3\downarrow}$ , which reduces to:  $2\Delta + h = 10(\frac{L}{2\pi})^2$  Similarly, the equation of the right boundary is:  $2\Delta - h = 8(\frac{L}{2\pi})^2$  in which  $\Delta = \mu_f - \mu_B$ .

### B. Strong-coupling limit

When the intensity of interaction approaches to infinity, any two particles can not stay in the same state. Thus there should be a much stronger excluding force than pauli principle. Because the energy of  $N$  particles ( $N - M$  bosons,  $M - M'$  spin up fermions,  $M'$  spin down fermions) is:  $H = E_0 - \mu_B \left( (N - M) + \frac{\mu_f}{\mu_B} M \right) - \frac{h}{2} (M - 2M')$ , when the direction of the magnetic field is along the spin up ( $h > 0$ ) direction, the spin down fermions can not appear in the ground state. Therefore, in region “BF1”, there are only fermions with spin up and bosons; And in region “F1”, there are  $N$  fermions with spin up. The same analysis hold when  $h < 0$ .

### C. In general case

We solve this model within the rules of the Young tableau ( $N - M \geq M - M' \geq M''$ ) by using Eqs.(30), thus the case of all the particles being fermions in ground state can not be gained here. And we only computed the case of  $h > 0$  because of symmetry. As shown in Fig.6, the phase boundary between “B” and “BF1” is the same as that in the two limit cases above, which intersects with the  $h$ -axis at  $h = 2\mu_B$ . The boundary between “BF1” and “BF1F2” can

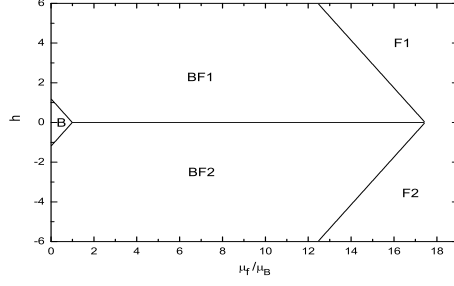


FIG. 5: The phase diagram of particles with strong coupling limit  $c \rightarrow \infty$ .

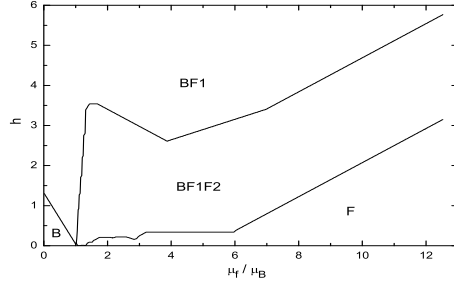


FIG. 6: A sketch of phase when interaction strength  $c = 1$

be divided into two parts from  $\mu_f/\mu_B \approx 1.5$ . The left part has some fluctuations which means adding a pair of spin up fermions in “BF1F2” side. The highest point ( $\mu_f/\mu_B \approx 1.5$ ) stands for 19 spin up and 1 spin down fermions. According the Young tableau, it can not add spin up fermion pair on the “19 up 1 down” state any more. Thus when  $\mu_f/\mu_b > 1.5$ , we can only add the spin down Fermi pair. This is the reason why there is a inflexion at  $\mu_f/\mu_B \approx 1.5$ . We figure out an outline of the spin singlet phase “F”, in which we can also see that there are some small fluctuations which stand for adding a pair of spin singlet fermions. It somewhat likes the free particles case. Because of pauli excluding principle, only two fermions with different spins can stay in the same state. Hence in the spin singlet phase, the pair numbers are  $1, 3, 5 \dots$  along the horizontal axis. From  $\mu_f/\mu_B \approx 6$  on, the pair number equals 14, which is the largest pair number we can get here.

## VI. SUMMARY AND DISCUSSION

In summery, we have explicitly derived the Bethe-ansatz equation for the model of one-dimensional Bose-Fermi mixture by means of QISM. We analyzed the properties of the ground state and the low-lying excitations on the basis of the Bethe-ansatz equations. We found that the ground state of this system is the state with merely bosons. The low-lying excitations were discussed extensively. The energy-momentum spectrum for three types of excitations, holon-particle, one fermion, two fermions, were plotted for  $c = 10.0$  and  $c = 1.0$ . We discussed the phase diagram of the ground state in the presence of external magnetic fields and chemical potential, from which we can know about the populations of boson and fermion at a given magnetic field and chemical potential.



# APPENDIX A: PERMUTATION MATRICES AND THE GENERATORS OF SU(3) LIE ALGEBRA

For BFF case the permutation matrix reads

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \quad (A1)$$

For FBF case it reads

$$P_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \quad (A2)$$

and for FFB case it reads

$$P_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad (A3)$$

The generators for SU(3) Lie algebra are given by:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{-(\alpha_1+\alpha_2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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